

LARGE CONFORMAL METRICS WITH PRESCRIBED SCALAR CURVATURE

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ABSTRACT. Let (M, g) be an n -dimensional compact Riemannian manifold. Let h be a smooth function on M and assume that it has a critical point $\xi \in M$ such that $h(\xi) = 0$ and which satisfies a suitable flatness assumption. We are interested in finding conformal metrics $g_\lambda = u_\lambda^{\frac{4}{n-2}} g$, with $u > 0$, whose scalar curvature is the prescribed function $h_\lambda := \lambda^2 + h$, where λ is a small parameter.

In the positive case, i.e. when the scalar curvature R_g is strictly positive, we find a family of “bubbling” metrics g_λ , where u_λ blows-up at the point ξ and approaches zero far from ξ as λ goes to zero.

In the general case, if in addition we assume that there exists a non-degenerate conformal metric $g_0 = u_0^{\frac{4}{n-2}} g$, with $u_0 > 0$, whose scalar curvature is equal to h , then there exists a bounded family of conformal metrics $g_{0,\lambda} = u_{0,\lambda}^{\frac{4}{n-2}} g$, with $u_{0,\lambda} > 0$, which satisfies $u_{0,\lambda} \rightarrow u_0$ uniformly as $\lambda \rightarrow 0$. Here, we build a second family of “bubbling” metrics g_λ , where u_λ blows-up at the point ξ and approaches u_0 far from ξ as λ goes to zero. In particular, this shows that this problem admits more than one solution.

1. INTRODUCTION

Let (M, g) be a smooth compact manifold without boundary of dimension $n \geq 3$. The prescribed scalar curvature problem (with conformal change of metric) is

given a function h on M does there exist a metric \tilde{g} conformal to g such that the scalar curvature of \tilde{g} equals h ?

Given a metric \tilde{g} conformal to g , i.e. $\tilde{g} = u^{\frac{4}{n-2}} g$ for some smooth conformal factor $u > 0$, this problem is equivalent to finding a solution to

$$-\Delta_g u + c(n)R_g u = c(n)h u^p, \quad u > 0 \quad \text{on } M,$$

where $\Delta_g = \operatorname{div}_g \nabla_g$ is the Laplace-Beltrami operator, $c(n) = \frac{n-2}{4(n-1)}$, $p = \frac{n+2}{n-2}$, and R_g denotes the scalar curvature associated to the metric g . Since it does not change the problem, we replace $c(n)h$ with h . We are then led to study the problem

$$(1.1) \quad -\Delta_g u + c(n)R_g u = h u^p, \quad u > 0 \quad \text{on } M,$$

We suppose that h is not constant, otherwise we would be in the special case of the Yamabe problem which has been completely solved in the works by Yamabe [Yam60], Trudinger [Tru68], Aubin [Aub76a], and Schoen [Sch84]. For this reason we can assume in (1.1) that R_g is a constant.

Date: July 3, 2017.

2010 Mathematics Subject Classification. 35J60, 53C21.

Key words and phrases. Prescribed scalar curvature, blow-up phenomena.

The first author has been supported by Fondi di Ateneo “Sapienza”. The second author has been supported by a public grant overseen by the French National Research Agency (ANR) as part of the “Investissements d’Avenir” program (reference: ANR-10-LABX-0098, LabEx SMP).

In the book [Aub98, Chapter 6], Aubin gives an exhaustive description of known results. Next, we briefly recall some of them.

- **The negative case, i.e. $R_g < 0$.**

A necessary condition for existence is that $\int_M h d\nu_g < 0$ (a more general result can be found in [KW75]).

When $h < 0$, (1.1) has a unique solution (see for instance [KW75, Aub76a]). The situation turns out to be more complicated when h vanishes somewhere on M or if it changes sign. When $\max_M h = 0$, Kazdan and Warner [KW75], Ouyang [Ouy91], Vázquez and Véron [VV91], and del Pino [dP94] proved the existence of a unique solution, provided that a lower bound on R_g , depending on the zero set of h , is satisfied. The general case was studied by Rauzy in [Rau95], who extended the previous results to the case when h changes sign. Letting $h^- := \min\{h, 0\}$ and $h^+ := \max\{h, 0\}$, the theorem proved in [Rau95] reads as follows.

Theorem 1.1. *Let $\mathcal{A} := \left\{ u \in H_g^1(M) \mid u \geq 0, u \not\equiv 0, \int_M h^- u d\nu_g = 0 \right\}$ and*

$$\Lambda_0 := \inf_{u \in \mathcal{A}} \frac{\int_M |\nabla u|^2 d\nu_g}{\int_M u^2 d\nu_g},$$

with $\Lambda_0 = +\infty$ if $\mathcal{A} = \emptyset$. There exists a constant $C(h) > 0$, depending only on $\frac{\min_M h^-}{\int_M h^- d\nu_g}$, such that if

$$(1.2) \quad -c(n)R_g < \Lambda_0 \quad \text{and} \quad \frac{\max_M h^+}{\int_M |h^-| d\nu_g} < C(h)$$

then (1.1) has a solution.

The dependence of the constant $C(h)$ on the function h^- can be found in [AB97]. An interesting feature is that if h changes sign then the uniqueness is not true anymore, as showed by Rauzy in [Rau96].

Theorem 1.2. *Assume (1.2) and let $\xi \in M$ such that $h(\xi) = \max_M h > 0$. If one of the following conditions hold:*

- (1) $6 \leq n \leq 9$ and $\Delta_g h(\xi) = 0$;
 - (2) $n \geq 10$, the manifold is not locally conformally flat, and $\Delta_g h(\xi) = \Delta_g^2 h(\xi) = 0$;
- then (1.1) admits at least two distinct solutions.*

- **The zero case, i.e. $R_g = 0$.**

Necessary conditions for existence are that h changes sign and $\int_M h d\nu_g < 0$. Some of the existence results proved by Escobar and Schoen [ES86], Aubin and Hebey [AH91], and Bismuth [Bis98] can be summarized as follows.

Theorem 1.3. *Let $\xi \in M$ such that $h(\xi) = \max_M h > 0$. If one of the following conditions hold:*

- (1) $3 \leq n \leq 5$ and all the derivatives of h at the point ξ up to order $n - 3$ vanish;

- (2) (M, g) is locally conformally flat, $n \geq 6$ and all the derivatives of h at the point ξ up to order $n - 3$ vanish;
- (3) the Weyl's tensor at ξ does not vanish, [$n = 6$ and $\Delta_g h(\xi) = 0$], or [$n \geq 7$ and $\Delta_g h(\xi) = \Delta_g^2 h(\xi) = 0$];
- then (1.1) has a solution

• **The positive case, i.e. $R_g > 0$.**

A necessary condition for existence is that $\max_M h > 0$. Some of the existence results proved by Escobar and Schoen [ES86], Aubin and Hebey [AH91], and Hebey and Vaugon [HV93] can be summarized as follows.

Theorem 1.4. *Assume that (M, g) is not conformal to the standard sphere (\mathbb{S}^n, g_0) . Let $\xi \in M$ such that $h(\xi) = \max_M h > 0$. If one of the following conditions hold:*

- (1) $n = 3$ or [$n \geq 4$, (M, g) is locally conformally flat, and all the derivatives of h at the point ξ up to order $n - 2$ vanish];
- (2) the Weyl's tensor at ξ does not vanish, [$n = 6$ and $\Delta_g h(\xi) = 0$], or [$n \geq 7$ and $\Delta_g h(\xi) = \Delta_g^2 h(\xi) = 0$];
- then (1.1) has a solution.

The prescribed scalar curvature problem on the standard sphere has also been largely studied. We refer the interested reader to [HV92, Li95, Li96].

In the rest of this paper, we focus our attention on the case $h_\lambda(x) := \lambda^2 + h(x)$ where $h \in C^2(M)$ and $\lambda > 0$ is a small parameter. Namely, we study the problem

$$(1.3) \quad -\Delta_g u + c(n)R_g u = (\lambda^2 + h)u^p, \quad u > 0 \quad \text{on } M.$$

In order to state our main results, let us introduce two assumptions. In our first theorem we assume that h satisfies the following *global* condition:

$$(1.4) \quad \begin{aligned} & \text{there exists a non-degenerate solution } u_0 \text{ to} \\ & -\Delta_g u_0 + c(n)R_g u_0 = h u_0^p, \quad u_0 > 0 \quad \text{on } M. \end{aligned}$$

The existence of a solution to (1.4) is guaranteed if h is as in Theorems 1.1, 1.3, or 1.4. The non-degeneracy condition is a delicate issue and it is discussed in Subsection 2.3. *It would be interesting to see if for "generic" functions h the solutions of (1.4) are non-degenerate.*

Under this assumption, it is clear that if λ is small enough then (1.3) has a solution $u_{0,\lambda} \in C^2(M)$ such that $\|u_{0,\lambda} - u_0\|_{C^2(M)} \rightarrow 0$ as $\lambda \rightarrow 0$.

In addition, the following *local* condition is assumed in both of our results:

There exist a point $\xi \in M$ and some real numbers $\gamma \geq 2$, $a_1, \dots, a_n \neq 0$, with $\sum_{i=1}^n a_i > 0$,

such that, in some geodesic normal coordinate system centered at ξ , we have

$$(1.5) \quad \begin{aligned} h(y) &= -\sum_{i=1}^n a_i |y_i|^\gamma + R(y) \quad \text{if } y \in B(0, r), \text{ for some } r > 0, \\ & \text{where } R \text{ satisfies } \lim_{y \rightarrow 0} R(y) |y|^{-\gamma} = 0. \end{aligned}$$

In particular, $h(\xi) = \nabla h(\xi) = 0$. The number γ is called *the order of flatness* of h at the point ξ . Observe that ξ cannot be a minimum point and that if all the a_i 's are positive then ξ is a local maximum point of h .

Let us now introduce the standard n -dimensional bubbles, which are defined via

$$U_{\mu,y}(x) = \mu^{-\frac{n-2}{2}} U \left(\frac{x-y}{\mu} \right), \quad \mu > 0, \quad y \in \mathbb{R}^n,$$

where $U(x) = \alpha_n \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}}$, $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$.

These functions are all the positive solutions to the critical problem (see [Aub76b, Tal76])

$$-\Delta U = U^p \quad \text{in } \mathbb{R}^n.$$

Our first result concerns the multiplicity of solutions to problem (1.3).

Theorem 1.5. *Assume that (M, g) is not conformal to the standard sphere (\mathbb{S}^n, g_0) , (1.4), and (1.5). If one of the following conditions hold:*

- (1) $3 \leq n \leq 5$ and ξ is a non-degenerate critical point of h , i.e. $\gamma = 2$;
- (2) $n = 6$ and $\gamma \in (2, 4)$;
- (3) $7 \leq n \leq 9$ and $\gamma = 4$;
- (4) $n \geq 10$, (M, g) is locally conformally flat, and $\gamma \in (\frac{n-2}{2}, \frac{n}{2})$;
- (5) $n \geq 10$, the Weyl's tensor at ξ does not vanish, and $\gamma \in (4, 4 + \epsilon)$ for some $\epsilon > 0$;

then, provided λ is small enough, there exists a solution u_λ to problem (1.3) which blows-up at the point ξ as $\lambda \rightarrow 0$. Moreover, as $\lambda \rightarrow 0$, we have

$$\left\| u_\lambda(x) - u_0(x) - \lambda^{-\frac{n-2}{2}} \mu_\lambda^{-\frac{n-2}{2}} U \left(\frac{d_g(x, \xi_\lambda)}{\mu_\lambda} \right) \right\|_{H_g^1(M)} \rightarrow 0,$$

where the concentration point $\xi_\lambda \rightarrow \xi$ and the concentration parameter $\mu_\lambda \rightarrow 0$ with a suitable rate with respect to λ , which depends on the order of flatness γ (see (2.5), (2.7), (2.8), (2.12)).

Finally, if $h \in C^\infty(M)$, then $\lambda^2 + h$ is the scalar curvature of a metric conformal to g .

This is the first multiplicity result in the zero and positive cases. In the negative case, it extends the results of Theorem 1.2 to locally conformally flat manifolds, to low-dimensional manifolds (i.e. $3 \leq n \leq 5$), to higher-dimensional manifolds (i.e. $6 \leq n \leq 9$) when the order of flatness at the maximum point ξ is at least 2, and to non-locally conformally manifolds when $n \geq 10$ and the order of flatness at the maximum point ξ is at least 4. Moreover, it also provides an accurate description of the profile of the solution as λ approaches zero.

Our second result concerns the existence of solutions to problem (1.3) in the positive case, without need of the global condition (1.4) on h .

Theorem 1.6. *Assume that (M, g) is not conformal to the standard sphere (\mathbb{S}^n, g_0) , $R_g > 0$, and (1.5). If one of the following conditions hold:*

- (1) $n = 3, 4, 5$ or $[n \geq 6$ and (M, g) is locally conformally flat] and $\gamma \in (n-2, n)$;
- (2) $n \geq 6$, the Weyl's tensor at ξ does not vanish, and $\gamma \in (4, n)$;

then, provided λ is small enough, there exists a solution u_λ to problem (1.3) which blows-up at the point ξ as $\lambda \rightarrow 0$. Moreover, as $\lambda \rightarrow 0$, we have

$$\left\| u_\lambda(x) - \lambda^{-\frac{n-2}{2}} \mu_\lambda^{-\frac{n-2}{2}} U \left(\frac{d_g(x, \xi_\lambda)}{\mu_\lambda} \right) \right\|_{H_g^1(M)} \rightarrow 0,$$

where the concentration point $\xi_\lambda \rightarrow \xi$ and the concentration parameter $\mu_\lambda \rightarrow 0$ with a suitable rate with respect to λ , which depends on the order of flatness γ (see (4.1)).

Finally, if $h \in C^\infty(M)$, then $\lambda^2 + h$ is the scalar curvature of a metric conformal to g .

Our results have been inspired by the recent papers by Borer, Galimberti, and Struwe [BGS15] and del Pino and Román [dPR15], where the authors studied the prescribed Gauss curvature problem on a surface of dimension 2 in the negative case. In particular, they built large conformal metrics with prescribed Gauss curvature κ , which exhibit a bubbling behavior around maximum points of κ at zero level. We also refer the reader to [dPMP05, dPMRW16], where equations closely related to (1.1) have been treated.

The proof of our results relies on a Lyapunov-Schmidt procedure (see for instance [BLR95, dPFM03]). To prove Theorem 1.5, we look for solutions to (1.3) which share a suitable bubbling profile close to the point ξ and the profile of the solution to the unperturbed problem (1.4) far from the point ξ . The accurate description of the ansatz is given in Section 2, which also contains a non-degeneracy result. The finite dimensional reduction is performed in Section 3, which also includes the proof of Theorem 1.5. All the technical estimates are postponed in the Appendix. In Section 4 we prove Theorem 1.6, which can be easily deduced by combining the results proved in Section 3 and in the Appendix with some recent results obtained by Esposito, Pistoia, and Vétois in [EPV14].

Acknowledgments. We would like to thank Manuel del Pino for bringing this problem to our attention, as well as for many helpful discussions we had with him. We also thank the anonymous referee for his or her careful reading and very useful comments.

2. THE APPROXIMATED SOLUTION

2.1. The ansatz. To build the approximated solution close to the point ξ we use some ideas introduced in [EP14, RV13]. The main order of the approximated solution close to the point ξ looks like the bubble

$$(2.1) \quad \lambda^{-\frac{n-2}{2}} \mathcal{U}_{t,\tau}(x) := \lambda^{-\frac{n-2}{2}} \mu^{-\frac{n-2}{2}} U \left(\frac{\exp_{\xi}^{-1}(x)}{\mu} - \tau \right) \quad \text{if } d_g(x, \xi) \leq r,$$

where the point $\tau \in \mathbb{R}^n$ depends on λ and the parameter $\mu = \mu_{\lambda}(t)$ satisfies

$$(2.2) \quad \mu = t\lambda^{\beta} \quad \text{for some } t > 0 \text{ and } \beta > 1.$$

The choice of β depends on n , on the geometry of the manifold at the point ξ , i.e. the Weyl's tensor at ξ and on the order of flatness of the function h at ξ , i.e. the number $\gamma := 2 + \alpha$ in (1.5).

Let us be more precise. Let $r \in (0, i_g(M))$ be fixed, where $i_g(M)$ is the injectivity radius of (M, g) , which is strictly positive since the manifold is compact. Let $\chi \in C^{\infty}(\mathbb{R})$ be a cut-off function such that $0 \leq \chi \leq 1$ in \mathbb{R} , $\chi = 1$ in $[-r/2, r/2]$ and $\chi = 0$ in $\mathbb{R} \setminus (-r, r)$. We denote by d_g the geodesic distance in (M, g) and by \exp_{ξ}^{-1} the associated geodesic coordinate system. We look for solutions of (1.3) of the form

$$(2.3) \quad u_{\lambda}(x) = u_0(x) + \lambda^{-\frac{n-2}{2}} \mathcal{W}_{t,\tau}(x) + \phi_{\lambda}(x),$$

where the definition of the blowing-up term $\mathcal{W}_{t,\tau}$ depends on the dimension of the manifold and also on its geometric properties. The higher order term ϕ_{λ} belongs to a suitable space which will be introduced in the next section. More precisely, $\mathcal{W}_{t,\tau}$ is defined in three different ways:

- **The case $n = 3, 4, 5$.**

It is enough to assume

$$(2.4) \quad \mathcal{W}_{t,\tau}(x) = \chi(d_g(x, \xi)) \mathcal{U}_{t,\tau}(x),$$

where $\mathcal{U}_{t,\tau}$ is defined in (2.1). The concentration parameter μ satisfies

$$(2.5) \quad \mu = t\lambda^{\frac{n+2}{2\alpha-n+6}}, \quad \text{with } t > 0 \text{ provided } 0 \leq \alpha < n - 2.$$

- **The cases $n \geq 10$ when $\text{Weyl}_g(\xi)$ is non-zero and $6 \leq n \leq 9$.**

It is necessary to correct the bubble $\mathcal{U}_{t,\tau}$ defined in (2.1) by adding a higher order term as in [EP14], namely

$$(2.6) \quad \mathcal{W}_{t,\tau}(x) = \chi(d_g(x, \xi)) (\mathcal{U}_{t,\tau}(x) + \mu^2 \mathcal{V}_{t,\tau}(x))$$

where

$$\mathcal{V}_{t,\tau}(x) = \mu^{-\frac{n-2}{2}} V \left(\frac{\exp_\xi^{-1}(x)}{\mu} - \tau \right) \quad \text{if } d_g(x, \xi) \leq r.$$

The choice of parameter μ depends on n . More precisely if $n \geq 10$ and the Weyl's tensor at ξ is non-zero we choose

$$(2.7) \quad \mu = t\lambda^{\frac{2}{\alpha-2}}, \quad \text{with } t > 0 \text{ provided } 2 < \alpha < \frac{2n}{n-2}.$$

If $6 \leq n \leq 9$ we choose

$$(2.8) \quad \mu = t\lambda^{\frac{n+2}{2\alpha-n+6}}, \quad \text{with } t > 0 \text{ provided } \frac{n-6}{2} < \alpha < \min \left\{ \frac{16}{n-2}, \frac{n^2-6n+16}{2(n-2)} \right\}.$$

The function V is defined as follows. If we write $u(x) = u(\exp_\xi^{-1}(x))$ for $x \in B_g(\xi, r)$ and $y = \exp_\xi(x) \in B(0, r)$, then a comparison between the conformal Laplacian $\mathcal{L}_g = -\Delta_g + c(n)R_g$ with the euclidean Laplacian shows that there is an error, which at main order looks like

$$(2.9) \quad \mathcal{L}_g u + \Delta u \sim +\frac{1}{3} \sum_{a,b,i,j=1}^n R_{iabj}(\xi) y_a y_b \partial_{ij}^2 u + \sum_{i,l,k=1}^n \partial_l \Gamma_{ii}^k(\xi) y_l \partial_k u + c(n)R_g(\xi)u.$$

Here R_{iabj} denotes the Riemann curvature tensor, Γ_{ij}^k the Christoffel's symbols and R_g the scalar curvature. This easily follows by standard properties of the exponential map, which imply

$$-\Delta_g u = -\Delta u - (g^{ij} - \delta^{ij}) \partial_{ij}^2 u + g^{ij} \Gamma_{ij}^k \partial_k u,$$

with

$$g^{ij}(y) = \delta^{ij}(y) - \frac{1}{3} R_{iabj}(\xi) y_a y_b + O(|y|^3) \quad \text{and} \quad g^{ij}(y) \Gamma_{ij}^k(y) = \partial_l \Gamma_{ii}^k(\xi) y_l + O(|y|^2).$$

To build our solution it shall be necessary to kill the R.H.S of (2.9) by adding to the bubble a higher order term V whose existence has been established in [EP14]. To be more precise, we need to remind (see [BE91]) that all the solutions to the linear problem

$$-\Delta v = pU^{p-1}v \quad \text{in } \mathbb{R}^n,$$

are linear combinations of the functions

$$(2.10) \quad Z_0(x) = x \cdot \nabla U(x) + \frac{n-2}{2} U(x), \quad Z_i(x) = \partial_i U(x), \quad i = 1, \dots, n.$$

The correction term V is built in the following Proposition (see Section 2.2 in [EP14]).

Proposition 2.1. *There exist $\nu(\xi) \in \mathbb{R}$ and a function $V \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ solution to*

$$-\Delta V - f'(U)V = - \sum_{a,b,i,j=1}^n \frac{1}{3} R_{iabj}(\xi) y_a y_b \partial_{ij}^2 U - \sum_{i,l,k=1}^n \partial_l \Gamma_{ii}^k(\xi) y_l \partial_k U - c(n) R_g(\xi) U + \nu(\xi) Z_0,$$

in \mathbb{R}^n , with

$$\int_{\mathbb{R}^n} V(y) Z^i(y) dy = 0, \quad i = 0, 1, \dots, n$$

and

$$|V(y)| + |y| |\partial_k V(y)| + |y|^2 |\partial_{ij}^2 V(y)| = O\left(\frac{1}{(1 + |y|^2)^{\frac{n-4}{2}}}\right), \quad y \in \mathbb{R}^n.$$

• **The case $n \geq 10$ when (M, g) is locally conformally flat.**

In this case it is necessary to perform a conformal change of metric as in [RV13]. Indeed, there exists a function $\Lambda_\xi \in C^\infty(M)$ such that the conformal metric $g_\xi = \Lambda_\xi^{\frac{4}{n-2}} g$ is flat in $B_g(\xi, r)$. The metric can be chosen so that $\Lambda_\xi(\xi) = 1$. Then, we choose

$$(2.11) \quad \mathcal{W}_{t,\tau}(x) = \chi(d_{g_\xi}(x, \xi)) \Lambda_\xi(x) \mathcal{U}_{t,\tau}(x),$$

where $\mathcal{U}_{t,\tau}$ is defined in (2.1) and the exponential map is taken with respect to the new metric g_ξ . In this case, the concentration parameter μ satisfies

$$(2.12) \quad \mu = t \lambda^{\frac{n+2}{2\alpha-n+6}}, \quad \text{with } t > 0 \text{ provided } \frac{n-6}{2} < \alpha < \frac{n^2-6n+16}{2(n-2)}.$$

2.2. The higher order term. Let us consider the Sobolev space $H_g^1(M)$ equipped with the scalar product

$$(u, v) = \int_M (\langle \nabla_g u, \nabla_g v \rangle_g + uv) d\nu_g,$$

and let $\|\cdot\|$ be the induced norm. Let us introduce the space where the higher order term ϕ_λ in (2.3) belongs to. Let Z_0, Z_1, \dots, Z_n be the functions introduced in (2.10). We define

$$Z_{i,t,\tau}(x) = \mu^{-\frac{n-2}{2}} \chi(d_{g_\xi}(x, \xi)) \Lambda_\xi(x) Z_i \left(\frac{\exp_\xi^{-1}(x)}{\mu} - \tau \right) \quad i = 0, 1, \dots, n,$$

where g_ξ and Λ_ξ are defined as in (2.11) and we also agree that $g_\xi \equiv g$, $\Lambda_\xi(x) \equiv 1$ if the ansatz is (2.4) or (2.6). Therefore, $\phi_\lambda \in H^\perp$ where

$$H^\perp := \left\{ \phi \in H_g^1(M) : \int_M \phi Z_{i,t,\tau} d\nu_g = 0 \text{ for any } i = 0, 1, \dots, n \right\}.$$

2.3. A non-degeneracy result. When the solution u_0 of (1.4) is a minimum point of the energy functional naturally associated with the problem, the non-degeneracy is not difficult to obtain as showed in Lemma 2.2. In the general case u_0 is a critical point of the energy of a min-max type and so the non-degeneracy is a more delicate issue. As far as we know there are no results in this direction.

Lemma 2.2. *Assume $R_g < 0$, $\max_M h = 0$, and the set $\{x \in M \mid h(x) = 0\}$ has empty interior set. Then the unique solution u_0 to (1.4) is non-degenerate, i.e. the linear problem*

$$-\Delta_g \psi + c(n)R_g \psi - ph(x)u_0^{p-1} \psi = 0 \quad \text{on } M,$$

admits only the trivial solution.

Proof. Del Pino in [dP94] proved that problem (1.4) has a unique solution, which is a minimum point of the energy functional

$$J(u) := \frac{1}{2} \int_M (|\nabla_g u|_g^2 + c(n)R_g u^2) d\nu_g - \frac{1}{p+1} \int_M h|u|^{p+1} d\nu_g.$$

Therefore the quadratic form

$$D^2 J(u_0)[\phi, \phi] = \int_M (|\nabla_g \phi|_g^2 + c(n)R_g \phi^2 - ph u_0^{p-1} \phi^2) d\nu_g, \quad \phi \in H_g^1(M)$$

is positive definite. In particular, the problem

$$(2.13) \quad -\Delta_g \phi_i + c(n)R_g \phi_i - ph u_0^{p-1} \phi_i = \lambda_i \phi_i \quad \text{on } M,$$

has a non-negative first eigenvalue λ_1 with associated eigenfunction $\phi_1 > 0$ on M .

If $\lambda_1 = 0$ then we test (1.4) against ϕ_1 and (2.13) against u_0 , we subtract and we get

$$(p-1) \int_M h u_0^p \phi_1 d\nu_g = 0,$$

which gives a contradiction because $h \neq 0$ a.e. in M and $u_0 > 0$ on M . \square

3. THE FINITE DIMENSIONAL REDUCTION

We are going to solve problem

$$(3.1) \quad \mathcal{L}_g u = (\lambda^2 + h) f(u) \quad \text{on } M,$$

where \mathcal{L}_g is the conformal Laplacian and $f(u) = (u^+)^p$, $u^+(x) := \max\{u(x), 0\}$, using a Ljapunov-Schmidt procedure. We rewrite (3.1) as

$$(3.2) \quad L(\phi_\lambda) = -E + (\lambda^2 + h)N(\phi_\lambda) \quad \text{on } M,$$

where setting

$$(3.3) \quad \mathcal{U}_\lambda(x) = \mathcal{U}_{\lambda,t,\tau}(x) := \lambda^{-\frac{n-2}{2}} \mathcal{W}_{t,\tau}(x) + u_0(x),$$

the linear operator $L(\cdot)$ is defined by

$$(3.4) \quad L(\phi) := \mathcal{L}_g \phi - (\lambda^2 + h) f'(\mathcal{U}_\lambda) \phi,$$

the error term is defined by

$$(3.5) \quad E := \mathcal{L}_g \mathcal{U}_\lambda - (\lambda^2 + h) f(\mathcal{U}_\lambda)$$

and the higher order term $N(\cdot)$ is defined by

$$N(\phi) := f(\mathcal{U}_\lambda + \phi) - f(\mathcal{U}_\lambda) - f'(\mathcal{U}_\lambda) \phi.$$

First of all, it is necessary to estimate the error term E .

Proposition 3.1. *Let $a, b \in \mathbb{R}_+$ be such that $0 < a < b$ and K be a compact set in \mathbb{R}^n . There exist positive numbers λ_0, C and $\epsilon > 0$ such that for any $\lambda \in (0, \lambda_0)$, for any $t \in [a, b]$ and for any point $\tau \in K$ we have*

$$\|E\|_{L^{\frac{2n}{n+2}}(M)} \leq C\lambda^{\frac{2(n+2)-\alpha(n-2)}{2}+\epsilon} \quad \text{if (2.7) holds}$$

or

$$\|E\|_{L^{\frac{2n}{n+2}}(M)} \leq C\lambda^{\frac{(n-2-\alpha)(n-2)}{2(2\alpha-n+6)}+\epsilon} \quad \text{if (2.5) or (2.8) or (2.12) hold.}$$

Proof. The proof is postponed in Subsection 5.1. \square

Then, we develop a solvability theory for the linearized operator L defined in (3.4) under suitable orthogonality conditions.

Proposition 3.2. *Let $a, b \in \mathbb{R}_+$ be fixed numbers such that $0 < a < b$ and K be a compact set in \mathbb{R}^n . There exist positive numbers λ_0 and C , such that for any $\lambda \in (0, \lambda_0)$, for any $t \in [a, b]$ and for any point $\tau \in K$, given $\ell \in L^{\frac{2n}{n+2}}(M)$ there is a unique function $\phi_\lambda = \phi_{\lambda,t,\tau}(\ell)$ and unique scalars $c_i, i = 0, \dots, n$ which solve the linear problem*

$$(3.6) \quad \begin{cases} L(\phi) = \ell + \sum_{i=0}^n c_i Z_{i,t,\tau} & \text{on } M, \\ \int_M \phi Z_{i,t,\tau} d\nu_g = 0, & \text{for all } i = 0, \dots, n. \end{cases}$$

Moreover,

$$(3.7) \quad \|\phi_\lambda\|_{H_g^1(M)} \leq C\|\ell\|_{L^{\frac{2n}{n+2}}(M)}.$$

Proof. The proof is postponed in Subsection 5.2. \square

Next, we reduce the problem to a finite-dimensional one by solving a non-linear problem.

Proposition 3.3. *Let $a, b \in \mathbb{R}_+$ be fixed numbers such that $0 < a < b$ and K be a compact set in \mathbb{R}^n . There exist positive numbers λ_0 and C , such that for any $\lambda \in (0, \lambda_0)$, for any $t \in [a, b]$ and for any point $\tau \in K$, there is a unique function $\phi_\lambda = \phi_{\lambda,t,\tau}$ and unique scalars $c_i, i = 0, \dots, n$ which solve the non-linear problem*

$$(3.8) \quad \begin{cases} L(\phi) = -E + (\lambda^2 + h)N(\phi) + \sum_{i=0}^n c_i Z_{i,t,\tau} & \text{on } M, \\ \int_M \phi Z_{i,t,\tau} d\nu_g = 0, & \text{for all } i = 0, \dots, n. \end{cases}$$

Moreover,

$$(3.9) \quad \|\phi_\lambda\|_{H_g^1(M)} \leq C\|E\|_{L^{\frac{2n}{n+2}}(M)}$$

and ϕ_λ is continuously differentiable with respect to t and τ .

Proof. The proof relies on standard arguments (see [EPV14]). \square

After Problem (3.8) has been solved, we find a solution to Problem (3.2) if we manage to adjust (t, τ) in such a way that

$$(3.10) \quad c_i(t, \tau) = 0, \quad \text{for all } i = 0, \dots, n.$$

This problem is indeed variational: it is equivalent to finding critical points of a function of t, τ . To see that let us introduce the energy functional J_λ defined on $H_g^1(M)$ by

$$J_\lambda(u) = \int_M \left(\frac{1}{2} |\nabla_g u|_g^2 + \frac{1}{2} c(n) R_g u^2 - \frac{\lambda^2}{p+1} (u^+)^p u + \frac{1}{p+1} h (u^+)^p u \right) d\nu_g.$$

An important fact is that the positive critical points of J_λ are solutions to (1.3). For any number $t > 0$ and any point $\tau \in \mathbb{R}^n$, we define the reduced energy

$$(3.11) \quad \mathcal{J}_\lambda(t, \tau) := J_\lambda(\mathcal{U}_\lambda + \phi_\lambda),$$

where $\mathcal{U}_\lambda = \mathcal{U}_{\lambda, t, \tau}$ is as in (3.3) and $\phi_\lambda = \phi_{\lambda, t, \tau}$ is given by Proposition 3.3. Critical points of \mathcal{J}_λ correspond to solutions of (3.10) for small λ , as the following result states.

Lemma 3.4. *The following properties hold:*

- (I) *There exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ if $(t_\lambda, \tau_\lambda)$ is a critical point of \mathcal{J}_λ then the function $u_\lambda = \mathcal{U}_\lambda + \phi_{\lambda, t_\lambda, \tau_\lambda}$ is a solution to (3.1).*
- (II) *Let $a, b \in \mathbb{R}_+$ be fixed numbers such that $0 < a < b$ and let K be a compact set in \mathbb{R}^n . There exists $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, we have:*

- (a) *if $n \geq 10$, $\text{Weyl}_g(\xi) \neq 0$, and (2.7) holds then*

$$(3.12) \quad \mathcal{J}_\lambda(t, \tau) = A_0 - \lambda^{\frac{2(n+2)-\alpha(n-2)}{\alpha-2}} \left(\underbrace{\left[A_1 |\text{Weyl}_g(\xi)|_g^2 t^4 - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy \right]}_{\Theta_1(t, \tau)} + o(1) \right),$$

C^1 -uniformly with respect to $t \in [a, b]$ and $\tau \in K$;

- (b) *if one of the following conditions is satisfied:*

- (i) $3 \leq n \leq 5$ and (2.5) holds;
 - (ii) $6 \leq n \leq 9$ and (2.8) holds;
 - (iii) $n \geq 10$, (M, g) is locally conformally flat, and (2.12) holds;
- then

$$(3.13) \quad \mathcal{J}_\lambda(t, \tau) = A_0 - \lambda^{\frac{(n-2-\alpha)(n-2)}{2\alpha-n+6}} \left(\underbrace{\left[A_3 u_0(\xi) t^{\frac{n-2}{2}} - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy \right]}_{\Theta_2(t, \tau)} + o(1) \right),$$

C^1 -uniformly with respect to $t \in [a, b]$ and $\tau \in K$.

Here, A_1, A_2 , and A_3 are constants only depending on n and

$$(3.14) \quad A_0 := \int_M \left(\frac{1}{2} |\nabla_g u_0|_g^2 + \frac{1}{2} c(n) R_g u_0^2 - \frac{\lambda^2}{p+1} u_0^{p+1} + \frac{1}{p+1} h u_0^{p+1} \right) d\nu$$

$$+ \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla U|^2 - \frac{1}{p+1} U^{p+1} \right) dy.$$

Proof. The proof of (I) is standard (see [EPV14]). The proof of (II) is postponed in Subsection 5.4. \square

The next result is essential to find solutions to (1.3).

Lemma 3.5. *There exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ if $(t_\lambda, \tau_\lambda)$ is a critical point of \mathcal{J}_λ then the function $u_\lambda = \mathcal{W}_\lambda + \phi_{\lambda, t_\lambda, \tau_\lambda}$ is a classical solution to (1.3).*

Proof. By Lemma 3.4 we deduce that u_λ solves (3.1). Arguing as in Appendix B of [Str08], one easily sees that $u_\lambda \in C^2(M)$.

It only remains to prove that $u_\lambda > 0$ on M . This is immediate in the positive case, i.e. $R_g > 0$, because the maximum principle holds. Let us consider the case $R_g \leq 0$.

We consider the set $\Omega_\lambda := \{x \in M \mid (u_\lambda - \lambda)^-(x) < 0\}$. Let $m_0 := \min_M u_0 > 0$. By the definition of u_λ we immediately get that for all λ sufficiently small $\phi_\lambda < -\frac{m_0}{2}$ in Ω_λ . Thus, since $\phi_\lambda \rightarrow 0$ in $L^2(M)$, we deduce $|\Omega_\lambda| \rightarrow 0$ as $\lambda \rightarrow 0$. Now, set $v := u_\lambda - \lambda$.

Testing (3.1) against v^- we get

$$\begin{aligned} \int_{\Omega_\lambda} |\nabla_g v^-|_g^2 d\nu_g + c(n)R_g \int_{\Omega_\lambda} (v^-)^2 d\nu_g - \int_{0 < u_\lambda < \lambda} (\lambda^2 + h)(u_\lambda^+)^{p-1} (v^-)^2 d\nu_g \\ + \lambda \left[c(n)R_g \int_{\Omega_\lambda} v^- d\nu_g - \int_{0 < u_\lambda < \lambda} (h + \lambda^2)(u_\lambda^+)^{p-1} v^- d\nu_g \right] = 0 \end{aligned}$$

Poincaré's inequality yields

$$\int_{\Omega_\lambda} |\nabla_g v^-|_g^2 d\nu_g \geq C(\Omega_\lambda) \int_{\Omega_\lambda} (v^-)^2 d\nu_g$$

where $C(\Omega_\lambda)$ is a positive constant approaching $+\infty$ as $|\Omega_\lambda|$ goes to zero.

On the other hand

$$\left| \int_{0 < u_\lambda < \lambda} (\lambda^2 + h)(u_\lambda^+)^{p-1} (v^-)^2 d\nu_g \right| \leq C\lambda^{p-1} \|v^-\|_{L^2(M)}^2,$$

and

$$\left| \int_{0 < u_\lambda < \lambda} (h + \lambda^2)(u_\lambda^+)^{p-1} v^- d\nu_g \right| \leq C\lambda^{p-1} \int_{\Omega_\lambda} |v^-| d\nu,$$

for some positive constant C not depending on λ . Collecting the previous computations we get

$$\underbrace{(C(\Omega_\lambda) + c(n)R_g - C\lambda^{p-1})}_{>0 \text{ if } \lambda \approx 0} \|v^-\|_{L^2(M)}^2 + \lambda \underbrace{(c(n)|R_g| - C\lambda^{p-1})}_{>0 \text{ if } \lambda \approx 0} \int_{\Omega_\lambda} |v^-| d\nu_g \leq 0$$

which implies $v^- = 0$ if λ is small enough. Since $u_\lambda \in C^2(M)$ we deduce that $u_\lambda \geq \lambda > 0$ in M and the claim is proved. \square

3.1. Proof of the main result. Theorem 1.5 is an immediate consequence of the more general result.

Theorem 3.6. *Assume (1.5) with $\gamma := 2 + \alpha$. If one of the following conditions hold:*

- (1) $n \geq 10$, the Weyl's tensor at ξ does not vanish, and $2 < \alpha < \frac{2n}{n-2}$;
- (2) $3 \leq n \leq 5$ and $0 \leq \alpha < n - 2$;
- (3) $6 \leq n \leq 9$ and $\frac{n-6}{2} < \alpha < \min \left\{ \frac{16}{n-2}, \frac{n^2-6n+16}{2(n-2)} \right\}$;

(4) $n \geq 10$, (M, g) is locally conformally flat, and $\frac{n-6}{2} < \alpha < \frac{n^2-6n+16}{2(n-2)}$; then, provided λ is small enough, there exists a solution to (1.3) which blows-up at the point ξ as $\lambda \rightarrow 0$.

Moreover, if $h \in C^\infty(M)$ then $\lambda^2 + h$ is the scalar curvature of a metric conformal to g .

Proof. We will show that the functions Θ_1 and Θ_2 , defined respectively in (3.12) and (3.13), have a non-degenerate critical point provided $\sum_{i=1}^n a_i > 0$ and $a_i \neq 0$ for any i . As a consequence, provided λ is small enough, the reduced energy \mathcal{J}_λ has a critical point and by Lemma 3.5 we deduce the existence of a classical solution to problem (1.3), which concludes the proof.

Without loss of generality, we can consider the function

$$\Theta(t, \tau) := t^\beta - t^\gamma \sum_{i=1}^n a_i \int_{\mathbb{R}^n} |y_i + \tau_i|^\gamma f(y) dy, \quad (t, \tau) \in (0, +\infty) \times \mathbb{R}^n,$$

where $\beta = 4$ or $\beta = \frac{n-2}{2}$, $\gamma = \alpha + 2$ and $f(y) = \frac{A}{(1+|y|^2)^n}$ for some positive constant A . It is immediate to check that, because $\sum_{i=1}^n a_i > 0$, this function has a critical point $(t_0, 0)$, where t_0 solves

$$\beta t^\beta = \mathbf{c}_1 \gamma t^\gamma \sum_{i=1}^n a_i, \quad \text{with } \mathbf{c}_1 := \int_{\mathbb{R}^n} |y_i|^\gamma f(y) dy \text{ not depending on } i,$$

Moreover it is non-degenerate. Indeed, a straightforward computation shows that

$$D^2\phi(t_0, 0) = \begin{pmatrix} \beta(\beta - \gamma)t_0^{\beta-2} & 0 & \dots & 0 \\ 0 & -\gamma(\gamma - 1)\mathbf{c}_2 t_0^\gamma a_1 & \dots & 0 \\ 0 & 0 & \dots & -\gamma(\gamma - 1)\mathbf{c}_2 t_0^\gamma a_n \end{pmatrix},$$

where $\mathbf{c}_2 := \int_{\mathbb{R}^n} |y_i|^{\gamma-2} f(y) dy$ does not depend on i , which is invertible because $\beta \neq \gamma$, $\beta > 0$, and $a_i \neq 0$ for any i . \square

4. THE POSITIVE CASE: PROOF OF THEOREM 1.6

In this section we find a solution to equation (1.3) in the positive case, i.e. $R_g > 0$ only assuming the local behavior (1.5) of the function h around the local maximum point ξ . We build solutions to problem (1.3) which blow-up at ξ as λ goes to zero, by combining the ideas developed by Esposito, Pistoia and Vétois [EPV14], the Ljapunov-Schmidt argument used in Section 3 and the estimates computed in the Appendix. We omit all the details of the proof because they can be found (up to minor modifications) in [EPV14] and in the Appendix. We only write the profile of the solutions we are looking for and the reduced energy whose critical points produce solutions to our problem.

4.1. The ansatz. Let us recall the construction of the main order term of the solution performed in [EPV14]. In case (M, g) is locally conformally flat, there exists a family $(g_\xi)_{\xi \in M}$ of smooth conformal metrics to g such that g_ξ is flat in the geodesic ball $B_\xi(r_0)$. In case (M, g) is not locally conformally flat, we fix $N > n$, and we find a family $(g_\xi)_{\xi \in M}$ of smooth conformal metrics to g such that

$$|\exp_\xi^* g_\xi|(y) = 1 + O(|y|^N)$$

C^1 -uniformly with respect to $\xi \in M$ and $y \in T_\xi M$, $|y| \ll 1$, where $|\exp_\xi^* g_\xi|$ is the determinant of g_ξ in geodesic normal coordinates of g_ξ around ξ . Such coordinates are said to be *conformal normal coordinates* of order N on the manifold. Here, the exponential map \exp_ξ^* is intended with respect to the metric g_ξ . For any $\xi \in M$, we let Λ_ξ be the smooth positive function on M such that $g_\xi = \Lambda_\xi^{\frac{4}{n-2}} g$. In both cases (locally conformally flat or not), the metric g_ξ can be chosen smooth with respect to ξ and such that $\Lambda_\xi(\xi) = 1$ and $\nabla \Lambda_\xi(\xi) = 0$. We let G_g and G_{g_ξ} be the respective Green's functions of L_g and L_{g_ξ} . Using the fact that $\Lambda_\xi(\xi) = 1$, we deduce

$$G_g(\cdot, \xi) = \Lambda_\xi(\cdot) G_{g_\xi}(\cdot, \xi).$$

We define

$$\mathcal{W}_{t,\tau}(x) = G_g(x, \xi) \widehat{\mathcal{W}}_{t,\tau}(x),$$

with

$$\widehat{\mathcal{W}}_{t,\tau}(x) := \begin{cases} \beta_n \lambda^{-\frac{(n-2)}{2}} \mu^{-\frac{n-2}{2}} d_{g_\xi}(x, \xi)^{n-2} U\left(\frac{d_{g_\xi}(x, \xi)}{\mu} - \tau\right) & \text{if } d_{g_\xi}(x, \xi) \leq r \\ \beta_n \lambda^{-\frac{(n-2)}{2}} \mu^{-\frac{n-2}{2}} r^{n-2} U\left(\frac{r_0}{\mu} - \tau\right) & \text{if } d_{g_\xi}(x, \xi) > r, \end{cases}$$

where $\beta_n = (n-2)\omega_{n-1}$, ω_{n-1} is the volume of the unit $(n-1)$ -sphere, $\tau \in \mathbb{R}^n$ and the concentration parameter $\mu = \mu_\lambda(t)$ with $t > 0$ is defined as (here $\alpha = \gamma - 2$, being γ the order of flatness of h at the point ξ)

$$(4.1) \quad \mu = \begin{cases} t\lambda^{\frac{2}{4+\alpha-n}} & \text{if } n = 3, 4, 5 \text{ or } [n \geq 6 \text{ and } (M, g) \text{ is l.c.f.}] \text{ with } n-4 < \alpha < n-2 \\ t\ell^{-1}(\lambda^2) & \text{if } n = 6 \text{ and } \text{Weyl}_g(\xi) \neq 0 \text{ with } 2 < \alpha < 4 \\ t\lambda^{\frac{2}{\alpha-2}} & \text{if } n \geq 7 \text{ and } \text{Weyl}_g(\xi) \neq 0 \text{ with } 2 < \alpha < n-2, \end{cases}$$

where the function $\ell(\mu) := -\mu^{2-\alpha} \ln \mu$ when μ is small. We look for a solution to (1.3) as $u_\lambda = \mathcal{W}_{t,\tau} + \phi_\lambda$, where the higher order term is found arguing as in Section 3.

4.2. The reduced energy. Combining Lemma 1 in [EPV14] and Lemma 5.2 in the Appendix, the reduced energy \mathcal{J}_λ introduced in (3.11) (where the term \mathcal{U}_λ is replaced by $\mathcal{W}_{t,\tau}$ and in particular $u_0 = 0$) reads as

(a) if $n = 3, 4, 5$ or $[n \geq 6 \text{ and } (M, g) \text{ is l.c.f.}]$ with $n-4 < \alpha < n-2$ and (4.1) holds then

$$\mathcal{J}_\lambda(t, \tau) = A_0 - \lambda^{\frac{(n-6-\alpha)(n-2)}{4+\alpha-n}} \left(A_3 \mathbf{m}(\xi) t^{n-2} - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o(1) \right),$$

where $\mathbf{m}(\xi) > 0$ is the *mass* at the point ξ ,

(b) if $n = 6$, $\text{Weyl}_g(\xi) \neq 0$, and (4.1) holds then

$$\mathcal{J}_\lambda(t, \tau) = A_0 - \left(-A_1 |\text{Weyl}_g(\xi)|_g^2 \frac{\mu_\lambda^4(t) \ln \mu_\lambda(t)}{\lambda^4} - A_2 \frac{\mu_\lambda^{2+\alpha}(t)}{\lambda^6} \sum_{i=1}^6 \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^6} dy + o(1) \right),$$

(c) if $n \geq 7$, $\text{Weyl}_g(\xi) \neq 0$ and (4.1) holds then

$$\mathcal{J}_\lambda(t, \tau) = A_0 - \lambda^{\frac{2(n+2)-\alpha(n-2)}{\alpha-2}} \left(A_1 |\text{Weyl}_g(\xi)|_g^2 t^4 - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o(1) \right),$$

C^1 -uniformly with respect to t in compact sets of $(0, +\infty)$ and τ in compact sets of \mathbb{R}^n . Here A_1 , A_2 and A_3 are constants only depending on n and A_0 depends only on n and λ .

5. APPENDIX

We recall the following useful lemma (see, for example, [Li98]).

Lemma 5.1. *For any $a > 0$ and $b \in \mathbb{R}$ we have*

$$|((a+b))^q - a^q| \leq \begin{cases} c(q) \min\{|b|^q, a^{q-1}|b|\} & \text{if } 0 < q < 1, \\ c(q) (|b|^q + a^{q-1}|b|) & \text{if } q \geq 1, \end{cases}$$

and

$$\left| ((a+b)^+)^{q+1} - a^{q+1} - (q+1)a^q b \right| \leq \begin{cases} c(q) \min\{|b|^{q+1}, a^{q-1}b^2\} & \text{if } 0 < q < 1, \\ c(q) (|b|^{q+1} + a^{q-1}b^2) & \text{if } q \geq 1. \end{cases}$$

5.1. Estimate of the error.

Proof of Lemma 3.1. We split the error (3.5) into

$$E = (-\Delta_g + c(n)R_g) \left[\lambda^{-\frac{n-2}{2}} \mathcal{W}_{t,\tau} + u_0 \right] - (\lambda^2 + h) \left[\lambda^{-\frac{n-2}{2}} \mathcal{W}_{t,\tau} + u_0 \right]^p = E_1 + E_2 + E_3,$$

where

$$\begin{aligned} E_1 &= \lambda^{-\frac{n-2}{2}} \left[-\Delta_g \mathcal{W}_{t,\tau} + c(n)R_g \mathcal{W}_{t,\tau} + \mathcal{W}_{t,\tau}^p \right], \\ E_2 &= -\lambda^{-\frac{n-2}{2}} \left[(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0)^p - \mathcal{W}_{t,\tau}^p \right], \\ E_3 &= \lambda^{-\frac{n+2}{2}} h \left[(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0)^p - (\lambda^{\frac{n-2}{2}} u_0)^p \right]. \end{aligned}$$

To estimate E_2 and E_3 we use the fact that the bubble $\mathcal{W}_{t,\tau}$ satisfies in the three cases

$$(5.1) \quad \mathcal{W}_{t,\tau}(\exp_\xi(y)) = O\left(\frac{\mu^{\frac{n-2}{2}}}{(\mu^2 + |y - \mu\tau|^2)^{\frac{n-2}{2}}} \right) \quad \text{if } |y - \xi| \leq r.$$

Indeed by (5.1) and Lemma 5.1 we immediately deduce that

$$\begin{aligned} \|E_2\|_{L^{\frac{2n}{n+2}}(M)} &= O\left(\lambda^{-\frac{n-2}{2}} \left\| \lambda^{\frac{n-2}{2}} u_0 \mathcal{W}_{t,\tau}^{p-1} \right\|_{L^{\frac{2n}{n+2}}(M)} \right) + O\left(\lambda^{-\frac{n-2}{2}} \left\| \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right\|_{L^{\frac{2n}{n+2}}(M)} \right) \\ &= O\left(\left\| \mathcal{W}_{t,\tau}^{p-1} \right\|_{L^{\frac{2n}{n+2}}(M)} \right) + O(\lambda^2), \end{aligned}$$

with

$$\left\| \mathcal{W}_{t,\tau}^{p-1} \right\|_{L^{\frac{2n}{n+2}}(M)} = \begin{cases} O\left(\mu^{\frac{n-2}{2}} \right) & \text{if } 3 \leq n \leq 5, \\ O\left(\mu^2 |\ln \mu|^{\frac{2}{3}} \right) & \text{if } n = 6, \\ O\left(\mu^2 \right) & \text{if } n \geq 7, \end{cases}$$

and

$$\begin{aligned} \|E_3\|_{L^{\frac{2n}{n+2}}(M)} &= O\left(\lambda^{-\frac{n+2}{2}} \left\| h \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} \mathcal{W}_{t,\tau} \right\|_{L^{\frac{2n}{n+2}}(M)}\right) + O\left(\lambda^{-\frac{n+2}{2}} \|h\mathcal{W}_{t,\tau}^p\|_{L^{\frac{2n}{n+2}}(M)}\right) \\ &= O\left(\lambda^{-\frac{n-2}{2}} \|h\mathcal{W}_{t,\tau}\|_{L^{\frac{2n}{n+2}}(M)}\right) + O\left(\lambda^{-\frac{n+2}{2}} \|h\mathcal{W}_{t,\tau}^p\|_{L^{\frac{2n}{n+2}}(M)}\right), \end{aligned}$$

with

$$\|h\mathcal{W}_{t,\tau}\|_{L^{\frac{2n}{n+2}}(M)} = \begin{cases} O\left(\mu^{\frac{n-2}{2}}\right) & \text{if } n < 10 + 2\alpha, \\ O\left(\mu^{\frac{n-2}{2}} |\ln \mu|^{\frac{n+2}{2n}}\right) & \text{if } n = 10 + 2\alpha, \\ O\left(\mu^{4+\alpha}\right) & \text{if } n > 10 + 2\alpha, \end{cases}$$

and

$$\|h\mathcal{W}_{t,\tau}^p\|_{L^{\frac{2n}{n+2}}(M)} = \begin{cases} O\left(\mu^{2+\alpha}\right) & \text{if } 2\alpha < n - 2, \\ O\left(\mu^{\frac{n+2}{2}} |\ln \mu|^{\frac{n+2}{2n}}\right) & \text{if } 2\alpha = n - 2, \\ O\left(\mu^{\frac{n+2}{2}}\right) & \text{if } 2\alpha > n - 2. \end{cases}$$

Now, let us estimate E_1 . In the first two cases, we argue exactly as in Lemma 3.1 in [EP14] and we deduce that

$$\|E_1\|_{L^{\frac{2n}{n+2}}(M)} = \begin{cases} O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right) & \text{if } 3 \leq n \leq 7, \\ O\left(\frac{\mu^3 |\ln \mu|^{\frac{5}{8}}}{\lambda^3}\right) & \text{if } n = 8, \\ O\left(\frac{\mu^3}{\lambda^{\frac{7}{2}}}\right) & \text{if } n = 9, \\ O\left(\frac{\mu^2 \frac{n+2}{n-2}}{\lambda^{\frac{n-2}{2}}}\right) & \text{if } n \geq 10, \end{cases}$$

In the third case, arguing exactly as in Lemma 7.1 of [RV13] we get

$$\|E_1\|_{L^{\frac{2n}{n+2}}(M)} = O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right).$$

Collecting all the previous estimates we get the claim. \square

5.2. The linear theory.

Proof of Lemma 3.2. We prove (3.7) by contradiction. If the statement were false, there would exist sequences $(\lambda_m)_{m \in \mathbb{N}}$, $(t_m)_{m \in \mathbb{N}}$, $(\tau_m)_{m \in \mathbb{N}}$ such that (up to subsequence) $\lambda_m \downarrow 0$, $\frac{\mu_m}{\lambda_m} \downarrow 0$, $t_m \rightarrow t_0 > 0$ and $\tau_m \rightarrow \tau_0 \in \mathbb{R}^n$ and functions ϕ_m, ℓ_m with $\|\phi_m\|_{H_g^1(M)} = 1$, $\|\ell_m\|_{L^{\frac{2n}{n+2}}} \rightarrow 0$, such that for scalars c_i^m one has

$$(5.2) \quad \begin{cases} L(\phi_m) = \ell_m + \sum_{i=0}^n c_i^m Z_{i,t_m,\tau_m} & \text{on } M, \\ \int_M \phi_m Z_{i,t_m,\tau_m} d\nu_g = 0, & \text{for all } i = 0, \dots, n. \end{cases}$$

We change variable setting $y = \frac{\exp_{\xi_m}^{-1}(x)}{\mu_m} - \tau_m$. We remark that $d_g(x, \xi) = |\exp_{\xi}^{-1}(x)|$ and we set

$$\tilde{\phi}_m(y) = \mu_m^{\frac{n-2}{2}} \chi(\mu_m |y + \tau_m|) \phi_m(\exp_{\xi_m}(\mu_m(y + \tau_m))) \quad y \in \mathbb{R}^n.$$

Since $\|\phi_m\|_{H_g^1(M)} = 1$, we deduce that the scaled function $(\tilde{\phi}_m)_m$ is bounded in $D^{1,2}(\mathbb{R}^n)$. Up to subsequence, $\tilde{\phi}_m$ converges weakly to a function $\tilde{\phi} \in D^{1,2}(\mathbb{R}^n)$ and thus in $L^{p+1}(\mathbb{R}^n)$ due to the continuity of the embedding of $D^{1,2}(\mathbb{R}^n)$ into $L^{p+1}(\mathbb{R}^n)$.

Step 1: We show that $c_i^m \rightarrow 0$ as $m \rightarrow \infty$ for all $i = 0, \dots, n$.

We test (5.2) against Z_{i,t_m,τ_m} . Integration by parts gives

$$(5.3) \quad \begin{aligned} \int_M \langle \nabla_g \phi_m, \nabla_g Z_{i,t_m,\tau_m} \rangle_g d\nu_g + \int_M [R_g - (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m})] \phi_m Z_{i,t_m,\tau_m} d\nu_g \\ = \int_M \ell_m Z_{i,t_m,\tau_m} d\nu_g + \sum_{j=0}^n c_i^m \int_M Z_{j,t_m,\tau_m} Z_{i,t_m,\tau_m} d\nu_g. \end{aligned}$$

Observe that

$$\left| \int_M \ell_m Z_{i,t_m,\tau_m} d\nu_g \right| \leq \|\ell_m\|_{L^{\frac{2n}{n+2}}(M)} \|Z_{i,t_m,\tau_m}\|_{L^{\frac{2n}{n-2}}(M)} = o(1).$$

By change of variables we have

$$c_j^m \int_M Z_{j,t_m,\tau_m} Z_{i,t_m,\tau_m} d\nu_g = c_j^m \int_{\mathbb{R}^n} Z_j Z_i dy + o(1) = c_j^m \delta_{ij} \int_{\mathbb{R}^n} Z_j^2 dy + o(1),$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

Writing $\tilde{h}(y) = h(\exp_{\xi_m}(\mu_m(y + \tau_m)))$, note also that

$$\begin{aligned} R_g \int_M \phi_m Z_{i,t_m,\tau_m} d\nu_g + \int_M h f'(\mathcal{U}_{\lambda_m}) \phi_m Z_{i,t_m,\tau_m} d\nu_g \\ = R_g \mu_m^2 \int_{\mathbb{R}^n} \tilde{\phi}_m Z_i dy + \int_{\mathbb{R}^n} \frac{\tilde{h}}{\lambda_m^2} \tilde{\phi}_m f'(U) Z_i dy + o(1). \end{aligned}$$

On the other hand, standard computations show that

$$\begin{aligned} \int_M \langle \nabla_g \phi_m, \nabla_g Z_{i,t_m,\tau_m} \rangle_g d\nu_g - \lambda_m^2 \int_M f'(\mathcal{U}_{\lambda_m}) \phi_m Z_{i,t_m,\tau_m} d\nu_g \\ = \int_{\mathbb{R}^n} \nabla \tilde{\phi}_m \cdot \nabla Z_i dy - \int_{\mathbb{R}^n} f'(U) \tilde{\phi}_m Z_i dy + o(1) \\ = - \int_{\mathbb{R}^n} (\Delta Z_i + f'(U) Z_i) \tilde{\phi}_m dy + o(1). \end{aligned}$$

Since Z_i satisfies $-\Delta Z_i = f'(U) Z_i$ in \mathbb{R}^n , passing to the limit into (5.3) yields

$$\sum_{j=0}^n \lim_{m \rightarrow \infty} c_i^m \delta_{ij} \int_{\mathbb{R}^n} Z_j^2 dy = o(1).$$

Hence $\lim_{m \rightarrow \infty} c_i^m = 0$, for all $i = 0, \dots, n$.

Step 2: We show that $\tilde{\phi} \equiv 0$.

Given any smooth function $\tilde{\psi}$ with compact support in \mathbb{R}^n we define ψ by the relation

$$\psi(x) = \mu_m^{-\frac{n-2}{2}} \chi(d_g(x, \xi)) \tilde{\psi} \left(\frac{\exp_{\xi_m}^{-1}(x)}{\mu_m} - \tau_m \right) \quad x \in M.$$

We test (5.2) against ψ . Integration by parts gives

$$(5.4) \quad \int_M \langle \nabla_g \phi_m, \nabla_g \psi \rangle_g d\nu_g + \int_M [R_g - (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m})] \phi_m \psi d\nu_g \\ = \int_M \ell_m \psi d\nu_g + \sum_{j=0}^n c_j^m \int_M Z_{j, t_m, \tau_m} \psi d\nu_g.$$

By **Step 1** it is easy to see that

$$(5.5) \quad \int_M \ell_m \psi d\nu_g + \sum_{j=0}^n c_j^m \int_M Z_{j, t_m, \tau_m} \psi d\nu_g \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

On the other hand, by the same arguments given in the proof of **Step 1**, we have

$$\int_M \langle \nabla_g \phi_m, \nabla_g \psi \rangle_g d\nu_g + \int_M [R_g - (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m})] \phi_m \psi d\nu_g \\ \rightarrow \int_{\mathbb{R}^n} \nabla \tilde{\phi} \cdot \nabla \tilde{\psi} dy - \int_{\mathbb{R}^n} f'(U) \tilde{\phi} \tilde{\psi} dy$$

as $m \rightarrow \infty$. Hence, passing to the limit into (5.4) and integrating by parts we get

$$- \int_{\mathbb{R}^n} (\Delta \tilde{\phi} + f'(U) \tilde{\phi}) \tilde{\psi} dy = 0, \quad \text{for all } \tilde{\psi} \in C_c^\infty(\mathbb{R}^n).$$

We conclude that $\tilde{\phi}$ is a solution in $D^{1,2}(\mathbb{R}^n)$ to $-\Delta v = f'(U)v$ in \mathbb{R}^n . Thus $\tilde{\phi} = \sum_{j=0}^n \alpha_j Z_j$, for certain scalars α_j . But

$$0 = \int_M \phi_m Z_{i, t_m, \tau_m} d\nu_g = \int_{\mathbb{R}^n} \tilde{\phi}_m Z_i d\nu_g \quad \text{for all } i = 0, \dots, n.$$

Passing to the limit we get $\int_{\mathbb{R}^n} \tilde{\phi} Z_i d\nu_g = 0$ for $i = 0, \dots, n$, which implies $\alpha_i = 0$ for all i .

Step 3: We show that, up to subsequence, $\phi_m \rightarrow 0$ in $H_g^1(M)$.

Since $(\phi_m)_m$ is bounded in $H_g^1(M)$, up to subsequence, ϕ_m converges weakly to a function $\phi \in H_g^1(M)$, and thus in $L^{p+1}(M)$ due to the continuity of the embedding of $H_g^1(M)$ into $L^{p+1}(M)$. Moreover, $\phi_m \rightarrow \phi$ strongly in $L^2(M)$.

We test equation (3.6) against a function $\psi \in H_g^1(M)$. Integration by parts gives (5.4). Once again, by **Step 1** it is easy to see that (5.5) holds. By weak convergence

$$\int_M \langle \nabla_g \phi_m, \nabla_g \psi \rangle_g d\nu_g + R_g \int_M \phi_m \psi d\nu_g \\ \rightarrow \int_M \langle \nabla_g \phi, \nabla_g \psi \rangle_g d\nu_g + R_g \int_M \phi \psi d\nu_g, \quad \text{as } m \rightarrow \infty.$$

Claim: $-\int_M (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m}) \phi_m \psi d\nu_g \rightarrow \int_M h f'(u_0) \phi \psi d\nu_g$ as $m \rightarrow \infty$.

Assuming the claim is true, passing to the limit into (5.4) gives

$$\int_M \langle \nabla_g \phi, \nabla_g \psi \rangle_g d\nu_g + R_g \int_M \phi \psi d\nu_g + \int_M h f'(u_0) \phi \psi d\nu_g = 0.$$

Elliptic estimates show that ϕ is a classical solution to $-\Delta_g \phi + R_g \phi = +h f'(u_0) \phi$ on M . Lemma 2.2 yields $\phi \equiv 0$.

Proof of the claim: Note that

$$\lambda_m^2 \int_M f'(\mathcal{U}_{\lambda_m}) \phi_m \psi d\nu_g = \lambda_m^2 \int_M [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g + \lambda_m^2 \int_M f'(u_0) \phi_m \psi d\nu_g.$$

We have

$$\left| \lambda_m^2 \int_M f'(u_0) \phi_m \psi d\nu_g \right| \leq \lambda_m^2 \|f'(u_0)\|_{L^\infty(M)} \|\phi_m\|_{L^2(M)} \|\psi\|_{L^2(M)} \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

We define

$$\tilde{\psi}(y) = \mu_m^{\frac{n-2}{2}} \chi(\mu_m |y + \tau_m|) \psi(\exp_{\xi_m}(\mu_m(y + \tau_m))) \quad y \in \mathbb{R}^n.$$

By (3.3) and change of variables we have

$$\begin{aligned} & \left| \lambda_m^2 \int_M [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g \right| \\ & \leq C \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^2} |\tilde{\phi}_m(y)| |\tilde{\psi}(y)| dy \\ & \leq C \left\| \frac{1}{(1 + |\cdot|^2)^2} \right\|_{L^{2-\epsilon \frac{(n-2)}{2}}(\mathbb{R}^n)} \|\tilde{\phi}_m\|_{L^{\frac{2n}{n-2} \frac{1}{1+\epsilon}}(\mathbb{R}^n)} \|\tilde{\psi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}, \end{aligned}$$

for $0 < \epsilon \ll 1$. Note that $\|\tilde{\psi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \|\psi\|_{L^{\frac{2n}{n-2}}(M)}$. By **Step 2** $\|\tilde{\phi}_m\|_{L^{\frac{2n}{n-2} \frac{1}{1+\epsilon}}(\mathbb{R}^n)} \rightarrow 0$ as $m \rightarrow \infty$, since $\frac{2n}{n-2} \frac{1}{1+\epsilon} < \frac{2n}{n-2}$. Thus

$$\lambda_m^2 \int_M f'(\mathcal{U}_{\lambda_m}) \phi_m \psi d\nu_g \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

On the other hand

$$\int_M h f'(\mathcal{U}_{\lambda_m}) \phi_m \psi d\nu_g = \int_M h [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g + \int_M h f'(u_0) \phi_m \psi d\nu_g.$$

Dominated convergence theorem yields

$$\int_M h f'(u_0) \phi_m \psi d\nu_g \rightarrow \int_M h f'(u_0) \phi \psi d\nu_g, \quad \text{as } m \rightarrow \infty.$$

By (3.3) and change of variables we have

$$\left| \int_M h [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g \right| \leq C \frac{\mu_m^2}{\lambda_m^2} \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^2} |\tilde{\phi}_m(y)| |\tilde{\psi}(y)| dy$$

We have

$$\begin{aligned} \frac{\mu_m^2}{\lambda_m^2} \int_{\mathbb{R}^n} \chi(\mu_m |y|) \frac{|y|^2}{(1+|y|^2)^2} |\tilde{\phi}_m(y)| |\tilde{\psi}(y)| dy \\ \leq C \frac{\mu_m^2}{\lambda_m^2} \left\| \frac{\chi(\mu_m |\cdot|)}{(1+|\cdot|^2)} \right\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \|\tilde{\phi}_m\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \|\tilde{\psi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ \leq C \frac{\mu_m^2}{\lambda_m^2} |\ln \mu_m| \|\tilde{\phi}_m\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \|\tilde{\psi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \end{aligned}$$

By **Step 2** and our choice of μ_m in terms of λ_m (see (2.2)) we conclude that

$$\left| \int_M h[f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g \right|, \quad \text{as } m \rightarrow \infty.$$

The claim is thus proved.

Step 4: We show that $\|\phi_m\|_{H_g^1(M)} \rightarrow 0$.

We take in (5.4) $\psi = \phi_m$. We get

$$\begin{aligned} (5.6) \quad \int_M |\nabla_g \phi_m|_g^2 d\nu_g + \int_M [R_g - (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m})] \phi_m^2 d\nu_g \\ = \int_M \ell_m \phi_m d\nu_g + \sum_{j=0}^n c_j^m \int_M Z_{j,t_m,\tau_m} \phi_m d\nu_g. \end{aligned}$$

By **Step 1–3**, passing to the limit into (5.6) gives

$$\lim_{m \rightarrow \infty} \int_M |\nabla_g \phi_m|_g^2 d\nu_g = 0.$$

Since $\phi_m \rightharpoonup 0$ in $H_g^1(M)$, we conclude

$$\|\phi_m\|_{H_g^1(M)} \rightarrow 0, \quad \text{as } m \rightarrow \infty,$$

which yields a contradiction with the fact that $\|\phi_m\|_{H_g^1(M)} = 1$. This concludes the proof of (3.7).

The existence and uniqueness of ϕ_λ solution to Problem 3.6 follows from the Fredholm alternative. This finishes the proof of Lemma 3.3. \square

5.3. The reduced energy.

Proof of Lemma 3.3. The result of Proposition 3.2 implies that the unique solution $\phi_\lambda = T_{t,\tau}(\ell)$ of (3.6) defines a continuous linear map $T_{t,\tau}$ from the space $L^{\frac{2n}{n+2}}(M)$ into $H_g^1(M)$. Moreover, a standard argument shows that the operator $T_{t,\tau}$ is continuously differentiable with respect to t and τ .

In terms of the operator $T_{t,\tau}$, Problem 3.8 becomes

$$\phi_\lambda = T_{t,\tau}(-E + (\lambda^2 + h)N(\phi_\lambda)) =: A(\phi_\lambda).$$

We define the space

$$H = \left\{ \phi \in H_g^1(M) \mid \int_M \phi Z_{i,t,\tau} d\nu_g = 0, \text{ for all } i = 0, \dots, n \right\}.$$

For any positive real number η , let us consider the region

$$\mathcal{F}_\eta \equiv \left\{ \phi \in H \mid \|\phi\|_{H_g^1(M)} \leq \eta \|E\|_{L^{\frac{2n}{n+2}}(M)} \right\}.$$

From (3.7), we get

$$\|A(\phi_\lambda)\|_{H_g^1(M)} \leq C \left(\|E\|_{L^{\frac{2n}{n+2}}(M)} + \|N(\phi_\lambda)\|_{L^{\frac{2n}{n+2}}(M)} \right).$$

Observe that

$$\|N(\phi_\lambda)\|_{L^{\frac{2n}{n+2}}(M)} \leq C \|\phi_\lambda\|_{L^{\frac{2n}{n-2}}(M)}^p \leq C \|\phi_\lambda\|_{H_g^1(M)},$$

and

$$\|N(\phi_1) - N(\phi_2)\|_{L^{\frac{2n}{n+2}}(M)} \leq C \eta^{p-1} \|E\|_{L^{\frac{2n}{n-2}}(M)}^{p-1} \|\phi_1 - \phi_2\|_{H_g^1(M)},$$

for $\phi_1, \phi_2 \in \mathcal{F}_\eta$. By (3.9), we get

$$\|A(\phi_\lambda)\|_{H_g^1(M)} \leq C \|E\|_{L^{\frac{2n}{n-2}}(M)} \left(\eta^p \|E\|_{L^{\frac{2n}{n-2}}(M)}^{p-1} + 1 \right),$$

and

$$\|A(\phi_1) - A(\phi_2)\|_{H_g^1(M)} \leq C \eta^{p-1} \|E\|_{L^{\frac{2n}{n-2}}(M)}^{p-1} \|\phi_1 - \phi_2\|_{H_g^1(M)},$$

for $\phi_1, \phi_2 \in \mathcal{F}_\eta$.

Since $p - 1 \in (0, 1)$ for $n \geq 3$ and $\|E\|_{L^{\frac{2n}{n-2}}(M)} \rightarrow 0$ as $\lambda \rightarrow 0$, it follows that if η is sufficiently large and λ_0 is small enough then A is a contraction map from \mathcal{F}_η into itself, and therefore a unique fixed point of A exists in this region.

Moreover, since A depends continuously (in the $L^{\frac{2n}{n-2}}$ -norm) on t, τ the fixed point characterization obviously yields so for the map $t, \tau \rightarrow \phi$. Moreover, standard computations give that the partial derivatives $\partial_t \phi, \partial_{\tau_i} \phi$, $i = 1, \dots, n$ exist and define continuous functions of t, τ . Besides, there exists a constant $C > 0$ such that for all $i = 1, \dots, n$

$$(5.7) \quad \|\partial_t \phi\|_{H_g^1(M)} + \|\partial_{\tau_i} \phi\|_{H_g^1(M)} \leq C \|E\|_{L^{\frac{2n}{n+2}}(M)} + \|\partial_t E\|_{L^{\frac{2n}{n+2}}(M)} + \|\partial_{\tau_i} E\|_{L^{\frac{2n}{n+2}}(M)}.$$

That concludes the proof. \square

5.4. The reduced energy. It is quite standard to prove that, as $\lambda \rightarrow 0$, $J_\lambda(\mathcal{U}_\lambda + \phi_\lambda) = J_\lambda(\mathcal{U}_\lambda) + h.o.t.$ C^1 -uniformly on compact sets of $(0, +\infty) \times \mathbb{R}^n$ (see [EPV14]). It only remains to compute $J_\lambda(\mathcal{U}_\lambda)$.

Lemma 5.2. *Let $a, b \in \mathbb{R}_+$ be fixed numbers such that $0 < a < b$ and let K be a compact set in \mathbb{R}^n . There exists a positive number λ_0 such that for any $\lambda \in (0, \lambda_0)$ the following expansions hold C^1 -uniformly with respect to $t \in [a, b]$ and $\tau \in K$:*

(a) *if $n \geq 10$, $|\text{Weyl}_g(\xi)| \neq 0$, and (2.5) holds, we have*

$$J_\lambda(\mathcal{U}_\lambda) = A_0 - \lambda^{\frac{2(n+2)-\alpha(n-2)}{\alpha-2}} \left[A_1 |\text{Weyl}_g(\xi)|_g^2 t^4 - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1 + |y|^2)^n} dy + o(1) \right],$$

(b) *if one of the following conditions is satisfied:*

- (i) $3 \leq n \leq 5$ and (2.5) holds;
- (ii) $6 \leq n \leq 9$ and (2.8) holds;
- (iii) $n \geq 10$, (M, g) is locally conformally flat, and (2.12) holds;

then

$$J_\lambda(\mathcal{U}_\lambda) = A_0 - \lambda^{\frac{(n-2-\alpha)(n-2)}{2\alpha-n+6}} \left[A_3 u_0(\xi) t^{\frac{n-2}{2}} - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o(1) \right].$$

Here A_1 , A_2 and A_3 are constants only depending on n and A_0 is defined in (3.14).

Proof. We prove the C^0 -estimate. The C^1 -estimate can be carried out in a similar way (see [EPV14]).

Let us first prove (a) and (ii) and (iii) of (b). It is useful to recall that

$$\alpha < \frac{2n}{n-2} < n-2 \text{ if } n \geq 10, \quad \alpha < \frac{n^2 - 6n + 16}{2(n-2)} < n-2 \text{ if } n \geq 6.$$

Observe that

$$\begin{aligned} J_\lambda(\mathcal{U}_\lambda) &= \underbrace{\frac{1}{2} \int_M |\nabla_g u_0|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g u_0^2 d\nu_g + \frac{1}{p+1} \int_M h u_0^{p+1} d\nu_g}_{\text{independent on } \mu \text{ and } \tau} \\ &+ \underbrace{\lambda^{-(n-2)} \left[\frac{1}{2} \int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g \mathcal{W}_{t,\tau}^2 d\nu_g - \frac{1}{p+1} \int_M \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right]}_{\text{leading term in case (a)}} \\ &+ \underbrace{\lambda^{-\frac{n-2}{2}} \left[\int_M \langle \nabla_g \mathcal{W}_{t,\tau}, \nabla_g u_0 \rangle d\nu_g + \int_M c(n) R_g \mathcal{W}_{t,\tau} u_0 d\nu_g \right] - \lambda^{-\frac{n-2}{2}} \int_M h \mathcal{W}_{t,\tau} u_0^p d\nu_g}_{=0} \\ &- \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ &\quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\ &+ \underbrace{\lambda^2 \frac{1}{p+1} \int_M u_0^{p+1} d\nu_g}_{\text{independent of } \mu \text{ and } \tau} \\ &- \underbrace{\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g}_{\text{leading term in case (b)}} \\ &- \lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \\ &+ \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ &\quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\ &+ \underbrace{\lambda^{-n} \frac{1}{p+1} \int_M h \mathcal{W}_{t,\tau}^{p+1} d\nu_g}_{\text{leading term in every case}} \end{aligned}$$

$$+ \lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g$$

Concerning the leading terms, we need to distinguish two cases. If the manifold is not locally conformally flat, by Lemma 3.1 in [EP14] we deduce

$$(5.8) \quad \lambda^{-(n-2)} \left[\frac{1}{2} \int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g \mathcal{W}_{t,\tau}^2 d\nu_g - \frac{1}{p+1} \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right] \\ = \begin{cases} \lambda^{-(n-2)} \left[A(n) - B(n) |\text{Weyl}_g(\xi)|_g^2 \mu^4 + o(\mu^4) \right] & \text{if } n \geq 7, \\ \lambda^{-(n-2)} \left[A(n) - B(n) |\text{Weyl}_g(\xi)|_g^2 \mu^4 |\ln \mu| + o(\mu^4 |\ln \mu|) \right] & \text{if } n = 6. \end{cases}$$

If the manifold is locally conformally flat, by Lemma 5.2 in [RV13] we get

$$\lambda^{-(n-2)} \left[\frac{1}{2} \left(\int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + c(n) R_g \mathcal{W}_{t,\tau}^2 \right) d\nu_g - \frac{1}{p+1} \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right] = A(n) + O\left(\frac{\mu^{n-2}}{\lambda^{n-2}}\right).$$

Here $A(n)$ and $B(n)$ are positive constants independent only depending on n . Moreover, straightforward computations lead to

$$(5.9) \quad \lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g = u_0(\xi) \frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \alpha_n^p \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} dy + o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right)$$

and

$$(5.10) \quad \lambda^{-n} \frac{1}{p+1} \int_M h \mathcal{W}_{t,\tau}^{p+1} d\nu_g = \frac{\mu^{2+\alpha}}{\lambda^n} \frac{\alpha_n^{p+1}}{p+1} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o\left(\frac{\mu^{2+\alpha}}{\lambda^n}\right),$$

because $\alpha < n - 2$.

Now, if $n \geq 10$ and the manifold is not locally conformally flat, we choose $\mu = t \lambda^{\frac{2}{\alpha-2}}$ so that the leading terms are (5.8) and (5.10), namely

$$\frac{\mu^4}{\lambda^{n-2}} \sim \frac{\mu^{2+\alpha}}{\lambda^n} \quad \text{and} \quad \frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} = o\left(\frac{\mu^4}{\lambda^{n-2}}\right).$$

On the other hand, if $6 \leq n \leq 9$ we choose $\mu = t \lambda^{\frac{n+2}{2\alpha-n+6}}$ so that the leading terms are (5.9) and (5.10), namely

$$\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \sim \frac{\mu^{2+\alpha}}{\lambda^n} \quad \text{and} \quad \frac{\mu^4}{\lambda^{n-2}} = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right), \quad \text{provided that } \alpha < \frac{16}{n-2}.$$

The higher order terms are estimated only taking into account that the bubble $\mathcal{W}_{t,\tau}$ satisfies (5.1).

A simple computation shows that

$$\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g = O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \lambda^2 \int_{B(0,r)} \frac{1}{|y - \mu\tau|^{n-2}} dy\right) = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right)$$

and

$$\lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g = O\left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}} \int_{B(0,r)} \frac{1}{|y - \mu\tau|^{n-\alpha}} dy\right) = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right).$$

If $n = 6$ then $p + 1 = 3$. It follows that

$$\begin{aligned} \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g = 0 \end{aligned}$$

and

$$\begin{aligned} \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g = 0. \end{aligned}$$

If $n \geq 7$, we get

$$\begin{aligned} & \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ & \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\ &= \lambda^{-(n-2)} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ & \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\ &+ \lambda^{-(n-2)} \frac{1}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ & \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\ &= o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right), \end{aligned}$$

because by Lemma 5.1

$$\begin{aligned} & \lambda^{-(n-2)} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ & \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\ &= \lambda^{-(n-2)} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) \right] d\nu_g \\ &- \lambda^{-(n-2)} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} d\nu_g - \lambda^{-(n-2)} \int_{B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \\ &= O \left(\lambda^{-(n-2)} \int_{B_g(\xi, \sqrt{\mu})} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} d\nu_g \right) + O \left(\lambda^{-(n-2)} \int_{B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^{p-1} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^2 d\nu_g \right) \\ &+ O \left(\lambda^{-(n-2)} \int_{B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \right) \end{aligned}$$

$$\begin{aligned}
&= O\left(\lambda^2 \int_{B_g(\xi, \sqrt{\mu})} u_0^{p+1} d\nu_g\right) + O\left(\mu^2 \int_{B(0, \sqrt{\mu})} \frac{1}{|y - \mu\tau|^4} dy\right) \\
&+ O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-6}{2}}} \int_{B(0, \sqrt{\mu})} \frac{1}{|y - \mu\tau|^{n-2}} dy\right) \\
&= O\left(\lambda^2 \mu^{\frac{n}{2}}\right) + O\left(\mu^{2+\frac{n-4}{2}}\right) + O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-6}{2}}} \mu\right) = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right),
\end{aligned}$$

since if μ is small enough, for any $q < n$, we have

$$(5.11) \quad \int_{B(0, \sqrt{\mu})} \frac{1}{|y - \mu\tau|^q} dy = \int_{B(-\mu\tau, \sqrt{\mu})} \frac{1}{|y|^q} dy \leq \int_{B(0, 2\sqrt{\mu})} \frac{1}{|y|^q} dy = O\left(\mu^{\frac{n-q}{2}}\right)$$

and

$$\begin{aligned}
&\lambda^{-(n-2)} \frac{1}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
&\quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
&= \frac{\lambda^{-(n-2)}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
&- \frac{\lambda^{-(n-2)}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^{p+1} d\nu_g - \lambda^{-(n-2)} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g \\
&= O\left(\lambda^{-(n-2)} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^2 \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} d\nu_g\right) + O\left(\frac{\lambda^{-(n-2)}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^{p+1} d\nu_g\right) \\
&+ O\left(\lambda^{-\frac{n-2}{2}} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^p d\nu_g\right) \\
&= O\left(\lambda^{-(n-4)} \mu^{\frac{n}{2}}\right) + O\left(-\lambda^{-(n-2)} \mu^{\frac{n}{2}}\right) + O\left(\frac{\mu^{\frac{n}{2}}}{\lambda^{\frac{n-2}{2}}}\right) = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right),
\end{aligned}$$

Here, we used the fact that

$$\frac{\mu^{\frac{n}{2}}}{\lambda^{n-2}} = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right),$$

and our choice

$$\begin{aligned}
\alpha &< \frac{n^2 - 6n + 16}{2(n-2)} && \text{if } 7 \leq n \leq 9, \\
\frac{n-6}{2} &< \alpha &< \frac{n^2 - 6n + 16}{2(n-2)} && \text{if } n \geq 10 \text{ and } (M, g) \text{ is locally conformally flat,} \\
\alpha &< \frac{2n}{n-2} && \text{if } n \geq 10 \text{ and } (M, g) \text{ is not locally conformally flat.}
\end{aligned}$$

In a similar way, if $n \geq 7$, we get

$$\begin{aligned}
& \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
&= \lambda^{-n} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
&+ \lambda^{-n} \frac{1}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
&= o \left(\frac{\mu^{2+\alpha}}{\lambda^n} \right),
\end{aligned}$$

because by Lemma 5.1

$$\begin{aligned}
& \lambda^{-n} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
&= \lambda^{-n} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) \right] d\nu_g \\
&- \lambda^{-n} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} h \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} d\nu_g - \lambda^{-n} \int_{B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \\
&= O \left(\lambda^{-n} \int_{B_g(\xi, \sqrt{\mu})} h \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} d\nu_g \right) + O \left(\lambda^{-n} \int_{B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^{p-1} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^2 d\nu_g \right) \\
&+ O \left(\lambda^{-n} \int_{B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \right) \\
&= O \left(\int_{B_g(\xi, \sqrt{\mu})} (d_g(x, \xi))^{\alpha+2} d\nu_g \right) + O \left(\frac{\mu^2}{\lambda^2} \int_{B(0, \sqrt{\mu})} \frac{|y|^{\alpha+2}}{|y - \mu\tau|^4} dy \right) \\
&+ O \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \int_{B(0, \sqrt{\mu})} \frac{|y|^{\alpha+2}}{|y - \mu\tau|^{n-2}} dy \right) \\
&= O \left(\mu^{\frac{\alpha+2+n}{2}} \right) + O \left(\frac{\mu^{\frac{\alpha+2+n}{2}}}{\lambda^2} \right) + O \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \mu^{\frac{\alpha+4}{2}} \right) = o \left(\frac{\mu^{2+\alpha}}{\lambda^n} \right)
\end{aligned}$$

Here, we used (5.11) and

$$\lambda^{-n} \frac{1}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right.$$

$$\begin{aligned}
& -(p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \Big] d\nu_g \\
= & \frac{\lambda^{-n}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
& - \frac{\lambda^{-n}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^{p+1} d\nu_g - \lambda^{-n} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g \\
= & O \left(\lambda^{-n} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^2 \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} d\nu_g \right) + O \left(\lambda^{-n} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right) \\
& + O \left(\lambda^{-\frac{n+2}{2}} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^p d\nu_g \right) \\
= & \begin{cases} O \left(\frac{\mu^{\frac{\alpha+2+n}{2}}}{\lambda^{n-2}} \right) & \text{if } \alpha < n-6 \\ O \left(\frac{\mu^{\frac{\alpha+2+n}{2}}}{\lambda^{n-2}} \right) |\ln \mu| & \text{if } \alpha = n-6 \\ O \left(\frac{\mu^{n-2}}{\lambda^{n-2}} \right) & \text{if } \alpha > n-6 \end{cases} + O \left(\frac{\mu^{\frac{\alpha+2+n}{2}}}{\lambda^n} \right) + O \left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}} \right) \\
= & o \left(\frac{\mu^{2+\alpha}}{\lambda^n} \right) \text{ because } \alpha < n-2.
\end{aligned}$$

Collecting the previous computations we get the result.

Let us now prove (i) of (b). Observe that

$$\begin{aligned}
& J_\lambda(\mathcal{U}_\lambda) \\
= & \underbrace{\frac{1}{2} \int_M |\nabla_g u_0|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g u_0^2 d\nu_g + \frac{1}{p+1} \int_M h u_0^{p+1} d\nu_g}_{\text{independent of } \mu \text{ and } \tau} \\
& + \lambda^{-(n-2)} \left[\frac{1}{2} \int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g \mathcal{W}_{t,\tau}^2 d\nu_g - \frac{1}{p+1} \int_M \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right] \\
& + \lambda^{-\frac{n-2}{2}} \underbrace{\left[\int_M \langle \nabla_g \mathcal{W}_{t,\tau}, \nabla_g u_0 \rangle + \int_M c(n) R_g \mathcal{W}_{t,\tau} u_0 \right] d\nu_g - \lambda^{-\frac{n-2}{2}} \int_M h \mathcal{W}_{t,\tau} u_0^p d\nu_g}_{=0} \\
& - \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
& + \underbrace{\lambda^2 \frac{1}{p+1} \int_M u_0^{p+1} d\nu_g}_{\text{independent of } \mu \text{ and } \tau} \\
& - \underbrace{\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g}_{\text{leading term}}
\end{aligned}$$

$$\begin{aligned}
& - \lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \\
& + \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
& + \underbrace{\lambda^{-n} \frac{1}{p+1} \int_M h \mathcal{W}_{t,\tau}^{p+1} d\nu_g}_{\text{leading term}} \\
& + \lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g
\end{aligned}$$

Concerning the leading terms, straightforward computations lead to

$$\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g = u_0(\xi) \frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \alpha_n^p \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} dy + o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right)$$

and

$$\lambda^{-n} \frac{1}{p+1} \int_M h \mathcal{W}_{t,\tau}^{p+1} d\nu_g = \frac{\mu^{2+\alpha}}{\lambda^n} \frac{\alpha_n^{p+1}}{p+1} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o\left(\frac{\mu^{2+\alpha}}{\lambda^n}\right),$$

because $\alpha < n - 2$.

The higher order terms are estimated as follows. By [MPV09], we deduce that

$$\begin{aligned}
& \lambda^{-(n-2)} \left[\frac{1}{2} \int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g \mathcal{W}_{t,\tau}^2 d\nu_g - \frac{1}{p+1} \int_M \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right] \\
& = \lambda^{-(n-2)} A(n) + \begin{cases} O\left(\frac{\mu}{\lambda}\right) & \text{if } n = 3 \\ O\left(\frac{\mu^2 |\ln \mu|}{\lambda^2}\right) & \text{if } n = 4 \\ O\left(\frac{\mu^2}{\lambda^3}\right) & \text{if } n = 5 \end{cases} \\
& = \lambda^{-(n-2)} A(n) + o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right),
\end{aligned}$$

where $A(n)$ is a constant that only depends on n . A simple computation shows that

$$\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g = O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \lambda^2 \int_{B_g(\xi,r)} \frac{1}{(d_g(x,\xi))^{n-2}} d\nu_g\right) = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right)$$

and

$$\begin{aligned} \lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g &= O \left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}} \int_{B_g(\xi,r)} \frac{1}{(d_g(x,\xi))^{n-\alpha}} d\nu_g \right) \\ &= \begin{cases} O \left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}} \right) & \text{if } \alpha > 0 \\ O \left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}} |\ln \mu| \right) & \text{if } \alpha = 0 \end{cases} \\ &= o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right). \end{aligned}$$

Finally, by using that

$$|(a+b)^{p+1} - a^{p+1} - b^{p+1} - (p+1)ab^p - (p+1)a^p b| \leq c(n) (a^2 b^{p-1} + a^{p-1} b^2),$$

which holds if $p \geq 2$ (this is true if $n = 3, 4, 5$) for any $a, b \geq 0$, we get

$$\begin{aligned} \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\ = O \left(\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^2 \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} d\nu_g \right) + O \left(\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^{p-1} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^2 d\nu_g \right) \\ = O \left(\lambda^{-(n-4)} \int_M \mathcal{W}_{t,\tau}^2 d\nu_g \right) + O \left(\int_M \mathcal{W}_{t,\tau}^{p-1} d\nu_g \right) \\ = \begin{cases} O(\lambda \mu) & \text{if } n = 3 \\ O(\mu^2 |\ln \mu|) & \text{if } n = 4 \\ O(\lambda^{-1} \mu^2) & \text{if } n = 5 \end{cases} + \begin{cases} O(\mu) & \text{if } n = 3 \\ O(\mu^2 |\ln \mu|) & \text{if } n = 4 \\ O(\mu^2) & \text{if } n = 5 \end{cases} \\ = o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right) \end{aligned}$$

and

$$\begin{aligned} \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\ \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\ = O \left(\lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^2 \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} d\nu_g \right) + O \left(\lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^{p-1} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^2 d\nu_g \right) \\ = O \left(\lambda^{-(n-2)} \int_M h \mathcal{W}_{t,\tau}^2 d\nu_g \right) + O \left(\lambda^{-2} \int_M h \mathcal{W}_{t,\tau}^{p-1} d\nu_g \right) \\ = O \left(\lambda^{-(n-2)} \mu^{n-2} \right) + O \left(\lambda^{-2} \mu^2 \right) \\ = o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right). \end{aligned}$$

Collecting the previous computations we get the result. □

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